Model 28 circular slide rule by Concise Ltd.

1. Section 1.5: exponential functions
2. Section 1.6: inverse functions and logarithms
The inverse function: reverse engineering

**Observation**

If we reverse the direction of the arrows, then the result might not be a function.

- Exactly one arrow departs from every point in $D$.
- Points in $C$ that are not in the range of $f$ are not hit by an arrow.
- Points in the range of $f$ may be hit by more than two arrows.
**Definition**

*A function \( f: D \to C \) is one-to-one if \( f(x_1) \neq f(x_2) \) for every \( x_1 \) and \( x_2 \in D \) with \( x_1 \neq x_2 \).*

- This is equivalent with: for all \( x_1 \) and \( x_2 \in D \) we have: if \( f(x_1) = f(x_2) \) then \( x_1 = x_2 \).
- For a one-to-one function every point in \( C \) is the end point of *at most* one arrow.
Example

The function $f(x) = 2x - 1$ is one-to-one.
The Horizontal line Test

If $f$ is one-to-one, then a horizontal line intersects the graph of $f$ in at most one point.
Example

The function \( f(x) = 2x^2 - 1 \) is not one-to-one.

- Notice that from \( f(x_1) = f(x_2) \) follows: \( x_1^2 = x_2^2 \), which does not imply \( x_1 = x_2 \).

- Observe that

  \[
  f(1) = 2 \cdot 1^2 - 1 = 1,
  \]

  and

  \[
  f(-1) = 2 \cdot (-1)^2 - 1 = 1,
  \]

  hence \( f(1) = f(-1) \).

- The graph of \( f \) does not satisfy the horizontal line test.

- One counterexample suffices.
The inverse function

**Theorem**

If \( f: D \to C \) is one-to-one, then reversing the arrows yields a function from the range of \( f \) to \( D \).

\[
f^{-1}: \text{range}(f) \to D
\]

- This function is called the **inverse of** \( f \), and is denoted as \( f^{-1} \).
Finding the inverse function

- If \( y = f(x) \), then \( x = f^{-1}(y) \).
- Finding the inverse means: solve the equation \( y = f(x) \) for \( x \).

**Example**

*Find the inverse of \( f(x) = 2x - 1 \).*
Let $y = f(x)$. Then $(x, y)$ lies on the graph of $f$.

From $y = f(x)$ follows $x = f^{-1}(y)$, so $(y, x)$ lies on the graph of $f^{-1}$.

The points $(x, y)$ and $(y, x)$ are reflected across the line $y = x$.

The graph of $f^{-1}$ and the graph of $f$ are symmetric with respect to the line $y = x$. 
Definition

The constant function \( c : \mathbb{R} \rightarrow \mathbb{R} \) assigns \( c \) to every \( x \in \mathbb{R} \).

\[
\begin{align*}
D = \mathbb{R}, & \quad C = \mathbb{R} \\
\end{align*}
\]
Definition

The identical map \( \text{id} : \mathbb{R} \rightarrow \mathbb{R} \) assigns \( x \) to every \( x \in \mathbb{R} \).

\[
D = \mathbb{R}, \quad C = \mathbb{R}
\]
A linear function $f: \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) = ax + b, \quad a \neq 0.$$
Definition

For every integer \( n \) we define

\[
x^n = \begin{cases} 
  1 & \text{if } n = 0, \\
  x \cdot x \cdot \ldots \cdot x & \text{if } n \geq 1, \\
  \frac{1}{x^{|n|}} & \text{if } n < 0.
\end{cases}
\]
**Definition**

For every positive integer $n$ we define the $\sqrt[n]{x} = x^{\frac{1}{n}}$ as the inverse of $f(x) = x^n$ where the domain of $f$ is assumed to be

- $[0, \infty)$ if $n$ is even,
- $\mathbb{R}$ if $n$ is odd.

![Graphs of square root functions](image)

$\sqrt{x} = \sqrt[4]{x}$

$\sqrt[4]{x} \neq \sqrt{x}$
## Definition

- For arbitrary fractions \( \frac{p}{q} \) (with \( p \) an integer and \( q \) a positive integer) we define
  \[
  x^{\frac{p}{q}} = \left( x^{\frac{1}{q}} \right)^p.
  \]

- If \( \alpha \in \mathbb{R} \) is not a fraction, then \( x^{\alpha} \) is defined by limits. This is beyond the scope of this course.

## Basic properties

For arbitrary \(^1\) \( x, y, \alpha \) and \( \beta \) we have

1. \( x^0 = 1 \)
2. \( 1^\alpha = 1 \)
3. \( x^\alpha y^\alpha = (x y)^\alpha \)
4. \( x^{\alpha+\beta} = x^\alpha x^\beta \)
5. \( x^{\alpha-\beta} = \frac{x^\alpha}{x^\beta} \)
6. \( (x^\alpha)^\beta = x^{\alpha\beta} \)

\(^1\) Some combinations of \( x, y, \alpha \) and \( \beta \) may not be defined.
Examples

- $3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8} = 3^{\frac{9}{5}} = \sqrt[5]{3^9}$

- $\frac{\sqrt{11}^3}{\sqrt{11}} = (\sqrt{11})^{3-1} = (\sqrt{11})^2 = 11$

- $(7\sqrt{2})^{\sqrt{2}} = 7^{\sqrt{2} \cdot \sqrt{2}} = 7^2 = 49$

- $7^\pi \cdot 8^\pi = (7 \cdot 8)^\pi = 56^\pi$

- $\left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{4^{\frac{1}{2}}}{9^{\frac{1}{2}}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$
  or $\left(\frac{4}{9}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{9}} = \sqrt{\left(\frac{2}{3}\right)^2} = \frac{2}{3}$
If I have 1000 Euro in a savings account and the bank gives 5% interest each year, what will be my savings after 5 years?

<table>
<thead>
<tr>
<th>Year</th>
<th>Savings (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
</tr>
<tr>
<td>1</td>
<td>1000 \cdot (1.05) = 1050.00</td>
</tr>
<tr>
<td>2</td>
<td>1000 \cdot (1.05)^2 = 1102.50</td>
</tr>
<tr>
<td>3</td>
<td>1000 \cdot (1.05)^3 = 1157.63</td>
</tr>
<tr>
<td>4</td>
<td>1000 \cdot (1.05)^4 = 1215.51</td>
</tr>
<tr>
<td>5</td>
<td>1000 \cdot (1.05)^5 = 1267.28</td>
</tr>
</tbody>
</table>

![Graph showing exponential growth of savings over 5 years](image-url)
If I have 1000 Euro in a savings account and the bank gives 5% interest each year, what will be my savings after 35 years?

<table>
<thead>
<tr>
<th>Year</th>
<th>Savings (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
</tr>
<tr>
<td>5</td>
<td>1000 \cdot (1.05)^5 = 1267.28</td>
</tr>
<tr>
<td>10</td>
<td>1000 \cdot (1.05)^{10} = 1628.89</td>
</tr>
<tr>
<td>15</td>
<td>1000 \cdot (1.05)^{15} = 2078.93</td>
</tr>
<tr>
<td>20</td>
<td>1000 \cdot (1.05)^{20} = 2653.3</td>
</tr>
<tr>
<td>25</td>
<td>1000 \cdot (1.05)^{25} = 3386.35</td>
</tr>
<tr>
<td>30</td>
<td>1000 \cdot (1.05)^{30} = 4321.94</td>
</tr>
<tr>
<td>35</td>
<td>1000 \cdot (1.05)^{35} = 5516.02</td>
</tr>
</tbody>
</table>
**Definition**

Let $a > 0$. The **exponential function** with base $a$ is $f(x) = a^x$. 

\[ y = 2^x \]
\[ y = 3^x \]
\[ y = 10^{-x} = 0.1^x \]
Exponential growth and decay

Definition

- If a quantity $y$ depends on time and $y$ is proportional to an exponential function, then we say that $y$ grows exponentially.
- If the base is less than 1 we say that $y$ decays exponentially.

- the human population (annual growth percentage $\approx 1.14\%$),
- carbon dating (the half-life of $^{14}\text{C}$ is approximately 5730 years),
- compound interest,
- Moore’s law: the number of transistors on integrated circuits doubles approximately every two years.

Exponential growth and decay

If $y$ grows exponentially, then there are constants $a$ and $y_0$ such that

$$y(x) = y_0 a^x.$$
The natural exponential function

- The derivative of an exponential function is proportional to the function itself.
- If \( f(x) = a^x \) then \( f'(x) = K a^x \) for some constant \( K \).
- There is one specific base value for which \( K = 1 \). This base is called \( e \) and is approximately \( e \approx 2.71828182845904523536028747135266249775724709 \ldots \)
- The function \( e^x \) is called the **natural exponential function**.
Exponential growth and decay

Let $a > 0$, then there is a constant $c \in \mathbb{R}$ such that

$$a = e^c.$$ 

For every $x$ the following holds:

$$a^x = (e^c)^x = e^{cx}$$

If $y$ grows exponentially, then there are constants $c$ and $y_0$ such that

$$y(x) = y_0 e^{cx}.$$ 

- If $c > 0$, then $a > 1$ hence $y$ is exponentially growing, and $c$ is called the **growth rate**.
- If $c < 0$, then $a < 1$ hence $y$ is exponentially decaying, and $c$ is called the **decay rate**.
- The constant $y_0$ is equal to $y(0)$, and is called the **initial value**.
Definition

The logarithm with base $a$ is the inverse of the exponential function with base $a$:

\[ y = a^x \quad \iff \quad x = \log_a y \]
Logarithms are exponents

\[ \log_a x \]

\[ \log_a y \]

\[ R \]

\[ R^+ = (0, \infty) \]

\[ a^x \]

\[ x \quad \text{and} \quad y \]

\[ \begin{align*}
\log_2 1 &= 0 \quad \text{because} \quad 2^0 = 1, \\
\log_2 2 &= 1 \quad \text{because} \quad 2^1 = 2, \\
\log_2 4 &= 2 \quad \text{because} \quad 2^2 = 4, \\
\log_{10} 1000 &= 3 \quad \text{because} \quad 10^3 = 1000, \\
\log_3 81 &= 4 \quad \text{because} \quad 3^4 = 81, \\
\log_9 81 &= 2 \quad \text{because} \quad 9^2 = 81, \\
\log_2 .25 &= -2 \quad \text{because} \quad 2^{-2} = \frac{1}{4} = .25.
\end{align*} \]
The graph of $y = \log_a x$ is obtained by reflecting the graph of $y = a^x$ across the diagonal line $y = x$. 
Logarithmic laws

- \( \log_a 1 = 0 \)

- \( \log_a a = 1 \)

- \( \log_a (x y) = \log_a x + \log_a y \)

- \( \log_a \frac{x}{y} = \log_a x - \log_a y \)

- \( \log_a \frac{1}{y} = -\log_a y \)

- \( \log_a (x^p) = p \log_a x \)
Logarithms with special base

- We write the logarithm with base 10 as $\log x$.
- We write the logarithm with base $e$ as $\ln x$.
- The logarithm with base $e$ is called the **natural logarithm**.